4/5/2014
Μιγαδικές

Έστω: $x:[\alpha,\beta]\to\mathbb{R}$, συνεχής τότε $\exists$ το $\int_\alpha^\beta x(s)ds$

↓

ολοκλήρωμα κατά Riemann

Αν $\phi:[\alpha,\beta]\to\mathbb{R}$, συνεχής δηλαδή $\phi(t)=x(t)+iy(t)$ με $x(t),y(t)$ συνεχείς

Άρα $\int_\alpha^\beta \phi(s)ds = \int_\alpha^\beta x(s)ds + \int_\alpha^\beta y(s)ds$

$\Rightarrow \int_\alpha^\beta (\kappa\phi(s)+\lambda\psi(s))ds = \kappa\int_\alpha^\beta \phi(s)ds + \lambda\int_\alpha^\beta \psi(s)ds$

- $\left|\int_\alpha^\beta \phi(s)ds\right| \le \int_\alpha^\beta |\phi(s)|ds$!

Απόδειξη

$t, s\in[\alpha,\beta]$

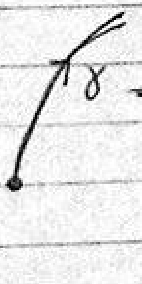
$(x(t),y(t)),\ (x(s),y(s))$

$\langle (x(t),y(t)),(x(s),y(s))\rangle = x(t)\cdot x(s) + y(t)\cdot y(s)$

$\le \sqrt{x^2(t)+y^2(t)}\cdot\sqrt{x^2(s)+y^2(s)}$

$x(t)\cdot\int_\alpha^\beta x + y(t)\cdot\int_\alpha^\beta y \le \sqrt{x^2(t)+y^2(t)}\cdot\int_\alpha^\beta\sqrt{x^2+y^2}$

$\left(\int_\alpha^\beta x\right)^2 + \left(\int_\alpha^\beta y\right)^2 \le \left(\int_\alpha^\beta \sqrt{x^2+y^2}\right)^2$ Ζητούμενο.



$\gamma \to \mathbb{C}$ συνεχής

$z(t),\ t\in[\alpha,\beta]$

$f(z(t))\cdot z'(t) \to$ συνεχής

Άρα το $\int_\alpha^\beta f(z(t))\cdot z'(t)\ \exists$

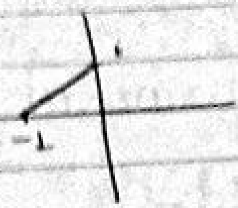
$\Downarrow$

Επικαμπύλιο ολοκλήρωμα

$\boxed{\int_\gamma f(z)dz = \int_\gamma f}$

$\int_\gamma (a f(z) + b g(z))dz = a\int_\gamma f(z) + b \int_\gamma g(z)dz$

Παράδειγμα
$f(z) = z^2 + \eta\mu z$

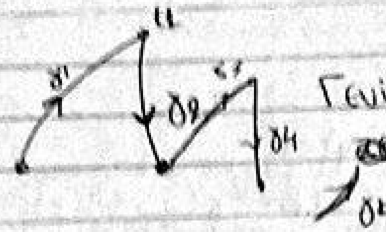


$\Rightarrow$ παραμετρική παράσταση
$z(t) = (1-t)(-1) + ti$
$= -1 + t + ti \quad t\in[0,1]$

Άρα $\int_\gamma f(z)dz = \int_0^1 [(-1+t+ti)^2 + \eta\mu(-1+t+ti)]\cdot(1+i)dt$
$(-1+t(1+i))^2$

$= \frac{(-1+t(1+i))^3}{3}\Big|_0^1 - \sigma\upsilon\nu(-1+t(1+i))\Big|_0^1$

$= \ldots$



Γενικά εαν έχω $\gamma_1+\gamma_2+\dots+\gamma_k$:
$z = z(t)$, $t\in[a,b]$
γ_i

t_1: $\exists z'(t_1,-)$
$\exists z'(t_1,+)$

$$\int_\gamma f(z)dz = \sum_{j=1}^{k} \int_{\gamma_j} f(z)dz$$

$C = \{\gamma_1,\gamma_2,\dots,\gamma_k\} = \gamma_1+\gamma_2+\dots+\gamma_k$
Τότε $\int_C f(z)dz = \sum_{j=1}^{k} \int_{\gamma_j} f(z)dz$

$f: \mathcal{U} \to \mathbb{C} \quad \lim_{\delta\to 0}\int_{\gamma_\delta} f(z)dz$

$\exists \rho>0: \overline{B(0,\rho)} \subseteq \mathcal{U}$
$\exists \delta_0: \delta<\delta_0 \Rightarrow \gamma_\delta \subseteq \overline{B(0,\rho)}$
$\exists M>0: |f(z)|\le M \quad \forall z\in\overline{B(0,\rho)}$

$$\left|\int_\gamma f(z)dz\right| \le M\cdot\int_{\gamma} |dz| = M\cdot l(\gamma)$$
$$= M\cdot 2\delta\sqrt{2} \to 0$$

$z = x+iy$

$dz = dx + i\,dy$

$|dz| = \sqrt{dx^2+dy^2}$

$x = x(t)$

$y = y(t)$

$l(\gamma) = \int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$

• $\int_\gamma f(z)dz = \int_a^b f(z(t))\cdot z'(t)\,dt = \int_a^b F'(z(t))\cdot z'(t)\,dt$

$= \int_a^b \frac{d}{dt} F(z(t))\,dt$

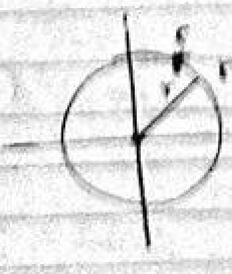
$= F(z(b)) - F(z(a))$

$= F(b) - F(a)$

$f: \to \mathbb{C}$ συνεχής

Η F είναι ορισμένη σε ανοιχτό του τόπο, τέτοια ώστε $F' = f$

$$\frac{F(z+h)-F(z)}{h}$$



re^{it}, $t\in[0,2\pi]$

$$\int_\gamma \frac{\log(z)}{z}\,dz$$

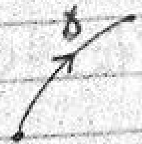
$\frac{1}{2}\log^2(z)$

Πρόταση: $g(z) = \sum_{\nu=0}^{+\infty} f_\nu(z)$, $z\in\gamma$

συνεχής

(1) f_ν $\to \mathbb{C}$

$$\int_\gamma \sum f_\nu(z)dz = \sum \int_\gamma f_\nu(z)dz$$

(2) γ  καρτεσιανό γινόμενο

$\times \alpha : Z \to \mathbb{C}$ συνεχής

$f(z, J)$

$G(J) = \int_\gamma f(z, J)\,dz \quad \forall J : Z \to \mathbb{C}$

<u>Συμπέρασμα:</u> G συνεχής

(3) $f_J(z, J) : Z \to \mathbb{C}$

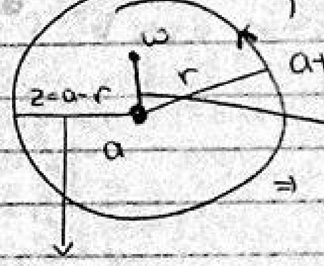
συνεχής

$G'(J) = \int_\gamma f_J(z, J)\,dz$

$$\frac{d}{dJ} \int_\gamma f(z, J)\,dz = \int_\gamma \frac{df}{dJ}(z, J)\,dz$$

$\nu \in \mathbb{Z} \quad \int_\gamma (z-a)^\nu dz = \begin{cases} 0, & \nu \neq -1 \\ \int_\gamma \frac{dz}{z-a}, & \nu = -1 \end{cases}$

$\circlearrowleft (+)$



$a + re^{it} \quad t \in [0, 2\pi]$

$\to (1-t)a + t\omega$

$= a + t(\omega - a)$

$\Rightarrow$

$1, \nu = -1$

↳ ανεξάρτητο του δρόμου καμπύλης.

~~$z-a$~~ -- ~~α~~ $z - a = a - r - a = -r < 0$

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\,dt = 2\pi i$$

$$\boxed{\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 1}$$

$\int_\gamma \frac{dz}{z-\omega} = \int_\gamma \frac{dz}{z - a + t(\omega - a)} \Rightarrow$ παντα $\exists$

↳ $= g(t)$ για το t μεταβλητή, $t \in [0, 1]$

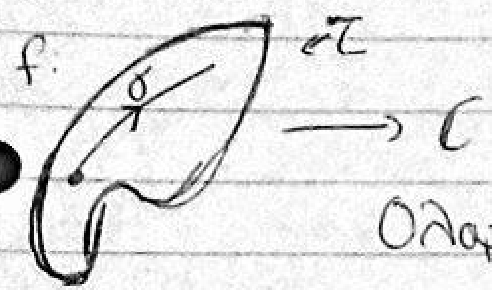
$g(0) = 2\pi i$, $g(1) = \int \frac{dz}{z-\omega}$

$$g'(t) = \int_\gamma \frac{w-a \, dz}{[(z-a)-t(w-a)]^2} = 0$$

$$\Rightarrow g(t) = c$$

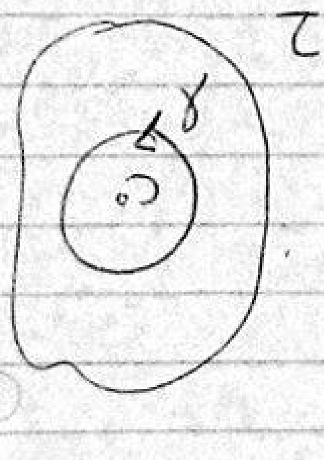
$$F(z) = \frac{-(w-a)}{(z-a)-t(w-a)}$$

Άρα $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w} = 1$



Ολομορφη με f' συνεχης

$$\int_\gamma \frac{f(z)}{z-\zeta} dz$$



$$f(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-\zeta}, \quad \zeta \in E\sigma(\gamma)$$

$\zeta \in E\xi(\gamma)$

$=0$